The Energy Components of the Gravitational Field

Erwin Schrödinger*
(From the II. Physics Institute of the Imperial and Royal University of Vienna
(Dated: November 22, 1917)

1. In the gravitational theory of general relativity, particularly for the manner of introducing the energy tensor of matter into the field equations, the 16 quantities t_σ^α, which Einstein designates as the energy components of the gravitational field[1], play a decisive role. If one uses a coordinate system for which

$$\sqrt{-g} = 1, \tag{1}$$

then one obtains for the t_σ^α the relatively simple expressions

$$\chi t_\sigma^\alpha = \frac{1}{2}\delta_\sigma^\alpha g^{\mu\nu}\Gamma_{\mu\beta}^\lambda\Gamma_{\nu\lambda}^\beta - g^{\mu\nu}\Gamma_{\mu\beta}^\alpha\Gamma_{\nu\sigma}^\beta, \tag{2}$$

where—as will always be the case in the following unless explicitly noted otherwise—the summation is to be carried out over indices occurring twice from 1 to 4. Notation:

$$\Gamma_{\mu\nu}^\lambda = -g^{\lambda\beta}\begin{bmatrix}\mu\nu\\ \beta\end{bmatrix} = = -\frac{1}{2}g^{\lambda\beta}\left[\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta}\right]; \tag{3}$$

the $g_{\mu\nu}$ are defined in the usual way through the expression for the square of the (four-dimensional) line element:

$$ds^2 = g_{\mu\nu}dx_\mu dx_\nu; \quad g_{\nu\mu} = g_{\mu\nu}, \quad g = \text{Det. of } g_{\mu\nu}. \tag{4}$$

The $g^{\mu\nu}$ are the adjugate, normalized first-order minors in the scheme of the $g_{\mu\nu}$. Finally,

$$\delta_\sigma^\alpha = g_{\mu\sigma}g^{\mu\alpha} = \begin{cases}0\\1\end{cases} \text{for } \begin{cases}\sigma \neq \alpha\\ \sigma = \alpha\end{cases} \tag{5}$$

and χ is (essentially) the gravitational constant.

The subject of this communication is the explicit calculation of the quantities t_σ^α in the vicinity of a stationary sphere of incompressible, gravitating fluid. The calculation is carried out exactly, based on the values of $g_{\mu\nu}$ determined by Schwarzschild[2], for a spatial coordinate system that differs only extremely slightly from a right-angled Cartesian one, indeed one might even label it as such. One must always specify the coordinate system in any calculation of the t_σ^μ, because these quantities do not form a tensor; for example, they do not vanish in all systems if they do so in a particular coordinate choice. The result reached in this special case—exact, identical vanishing of all t_σ^μ in the chosen reference frame—seems to me nevertheless so astonishing that I believe it should be brought to general discussion.

2. Schwarzschild finds in loc. cit. for the square of the line element

$$ds^2 = (1-\alpha/R)dt^2 - \frac{dR^2}{1-\alpha/R} - R^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) \tag{6}$$

with the abbreviation

$$R = (r^3 + \rho)^{1/3}. \tag{7}$$

r, ϑ, ϕ, t are ordinary polar coordinates and time; α and ρ are integration constants, which depend on the density and radius of the gravitating sphere and are, in reality, always extraordinarily small compared to all considered values of r or r^3.

In (6) we introduce new coordinates x_1, x_2, x_3, x_4 through the equations

$$\begin{aligned} x_1 &= R\sin\vartheta\cos\phi\\ x_2 &= R\sin\vartheta\sin\phi\\ x_3 &= R\cos\vartheta,\\ x_4 &= t. \end{aligned} \tag{8}$$

From this, it follows in the usual way:

$$R^2 = x_1^2 + x_2^2 + x_3^2, \tag{9}$$

further,

$$\begin{aligned} dt &= dx_4,\\ dR^2 &= \frac{x_\mu x_\nu}{R^2}dx_\mu dx_\nu\\ R^2(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) &= dx_1^2 + dx_2^2 + dx_3^2 - dR^2\\ &= \left(\delta_{\mu\nu} - \frac{x_\mu x_\nu}{R^2}\right)dx_\mu dx_\nu \end{aligned} \tag{10}$$

$$\left[\mu,\nu = 1,2,3; \quad \delta_{\mu\nu}\begin{cases}=0\\=1\end{cases} \text{for } \mu\begin{cases}\neq\\=\end{cases}\nu\right].$$

(Here we temporarily use our summation notation somewhat inconsistently for sums that run only from 1 to 3!)

By substituting (10) into (6), one obtains the line element in the new coordinates

$$ds^2 = (1-\alpha/R)dx_4^2 - \left[\delta_{\mu\nu} + \frac{\alpha x_\mu x_\nu}{R^3(1-\alpha/R)}\right]dx_\mu dx_\nu, \tag{11}$$

---

* Physik. Zeitschr. XIX. (1918). Translation by V. T. Toth, utilizing the technology of the GPT-4 o1-preview (2024) large language model by OpenAI Inc.

[1] A. Einstein, *The Foundation of the General Theory of Relativity* (J. A. Barth 1916), pp. 45 ff.

[2] Schwarzschild, Berl. Ber. 1916, p. 424.

from which one reads off from (4):

$$g_{\mu\nu} = \left[\delta_{\mu\nu} + \frac{\alpha x_\mu x_\nu}{R^3(1-\alpha/R)}\right] \text{ for } \mu,\nu = 1,2,3.$$
$$g_{14} = g_{24} = g_{34} = 0; \qquad g_{44} = 1 - \alpha/R. \tag{12}$$

The letter R here and henceforth is to be understood as an abbreviation for

$$R = +\sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{13}$$

In order to apply equations (2) and (3) to the calculation of the t_σ^α, we must first show that the chosen reference system satisfies the condition (1). First, one obtains on a coordinate axis, say the x_1-axis ($x_2 = x_3 = 0$), for the fundamental tensor (12) the simple scheme:

$$g_{\mu\nu} = \begin{bmatrix} -\frac{R}{R-\alpha} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{R-\alpha}{R} \end{bmatrix} \tag{14}$$

For later use, we note immediately the scheme of the contravariant fundamental tensor at a point on the x_1-axis

$$|g^{\mu\nu}| = \begin{bmatrix} -\frac{R-\alpha}{R} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{R}{R-\alpha} \end{bmatrix}. \tag{15}$$

From (14) it follows, in view of the spherical symmetry of the field, that equation (1) is satisfied everywhere; for any arbitrary point can be relocated onto the x_1-axis by a transformation of determinant +1 (spatial rotation). Equations (2) and (3) are thus applicable.

The transformation just mentioned is, as is well known, linear. Under linear transformations, however, the quantities t_σ^α have tensorial covariance (which is easy to show); they possess, at any rate, linear homogeneous transformation formulas. Therefore, it will suffice to compute these quantities only at an arbitrary point on the x_1-axis (which greatly simplifies the calculation). Since it will be shown that they all vanish identically at such a point, we may conclude that they vanish identically everywhere.

From (12) one easily recognizes that, for a point on the x_1-axis, among the 40 quantities $\frac{\partial g_{\mu\nu}}{\partial x_\sigma}$, only a few differ from zero. For these, a straightforward calculation yields:

$$\left.\begin{aligned} &\frac{\partial g_{11}}{\partial x_1} = \frac{\alpha}{(R-\alpha)^2} \\ &\frac{\partial g_{12}}{\partial x_2} = \frac{\partial g_{13}}{\partial x_3} = -\frac{\alpha}{R(R-\alpha)} \\ &\frac{\partial g_{44}}{\partial x_1} = \frac{\alpha}{R^2} \\ &\text{All others} = 0. \end{aligned}\right\} \tag{17}$$

For the $\Gamma_{\mu\nu}^\lambda$ one then obtains from (3) and (15)

$$\Gamma_{\mu\nu}^\lambda = -g^{\lambda\lambda}\begin{bmatrix}\mu\nu \\ \lambda\end{bmatrix} = -\frac{1}{2}g^{\lambda\lambda}\left[\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda}\right]. \tag{18}$$

(That no summation over the index λ is to be performed may be indicated by the round brackets.) We thus need to examine the 40 quantities $\begin{bmatrix}\mu\nu \\ \lambda\end{bmatrix}$ to determine which of them are non-zero based on (17).

A. μ, ν, λ "spatial" (i.e., $= 1, 2, 3$).

1. $\mu \neq \nu$.

a) $\mu \neq \lambda \neq \nu$. (3 quantities). They vanish since among (17) there is no term with three different indices.

b) $\lambda = \mu$. (6 quantities).

$$\left(\begin{bmatrix}\mu\nu \\ \mu\end{bmatrix}\right) = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x_\nu}\right).$$

They vanish, since in (17) there are no such quantities with only spatial indices.

2. $\mu = \nu$.

a) $\lambda \neq \mu$. (6 quantities).

Non-zero when $\lambda = 1$, $\mu = 2, 3$. Specifically:

$$\begin{bmatrix}22 \\ 1\end{bmatrix} = \begin{bmatrix}33 \\ 1\end{bmatrix} = -\frac{\alpha}{R(R-\alpha)}.$$

b) $\lambda = \mu$. (3 quantities).

$$\left(\begin{bmatrix}\mu\mu \\ \mu\end{bmatrix}\right) = \frac{1}{2}\left(\frac{\partial g_{\mu\mu}}{\partial x_\mu}\right).$$

Non-zero for $\mu = 1$, specifically:

$$\begin{bmatrix}11 \\ 1\end{bmatrix} = \frac{1}{2}\frac{\alpha}{(R-\alpha)^2}.$$

B. One index equals 4. ($6 + 9 = 15$ quantities).

In each term, either differentiation with respect to x_4 occurs, or one of the quantities g_{14}, g_{24}, g_{34} is differentiated. Therefore, these 15 quantities vanish.

C. Two indices equal 4. ($3 + 3 = 6$ quantities).

Non-zero are apparently only

$$\begin{bmatrix}41 \\ 4\end{bmatrix} = -\begin{bmatrix}44 \\ 1\end{bmatrix} = \frac{1}{2}\frac{\alpha}{R^2}.$$

D. All three indices equal 4. (1 quantity).

Vanishes. –

Based on this survey, one finds from (18) in consideration of (15):

$$\left.\begin{array}{l}\Gamma^1_{22} = \Gamma^1_{33} = -\dfrac{\alpha}{R^2},\\ \Gamma^1_{11} = -\Gamma^4_{41} = \dfrac{1}{2}\dfrac{\alpha}{R(R-\alpha)},\\ \Gamma^1_{44} = -\dfrac{1}{2}\dfrac{\alpha(R-\alpha)}{R^3},\\ \text{All others} = 0.-\end{array}\right\} \tag{19}$$

The expressions (2), which we now have to form, can be written with the abbreviation

$$A^\alpha_\sigma = g^{\mu\nu}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\sigma} \tag{20}$$

as follows:

$$\chi t^\alpha_\sigma = \frac{1}{2}\delta^\alpha_\sigma A^\lambda_\lambda - A^\alpha_\sigma. \tag{21}$$

We compute A^α_σ. Due to (15), all terms in (20) with $\mu \neq \nu$ vanish. Writing somewhat more explicitly:

$$\begin{aligned}A^\alpha_\sigma &= g^{11}\Gamma^\alpha_{1\beta}\Gamma^\beta_{1\sigma} + g^{22}\Gamma^\alpha_{2\beta}\Gamma^\beta_{2\sigma}\\ &+ g^{33}\Gamma^\alpha_{3\beta}\Gamma^\beta_{3\sigma} + g^{44}\Gamma^\alpha_{4\beta}\Gamma^\beta_{4\sigma},\end{aligned} \tag{22}$$

one recognizes that the terms grouped in the 2nd and 3rd terms also vanish individually due to (19): if $\beta = 2$ or 3 because of the third factor, otherwise because of the second factor. Remaining is

$$A^\alpha_\sigma = g^{11}\Gamma^\alpha_{1\beta}\Gamma^\beta_{1\sigma} + g^{44}\Gamma^\alpha_{4\beta}\Gamma^\beta_{4\sigma}. \tag{23}$$

From this, it is immediately evident that all A^α_σ containing the index 2 or 3 vanish. Thus, only the following remain to be examined:

$$A^4_1, \quad A^1_4, \quad A^1_1, \quad A^4_4. \tag{24}$$

In the expressions concerned, all terms in which $\beta = 2, 3$ fall away, as well as those in which a Γ quantity contains the index 4 once or three times. From this, it follows

$$A^4_1 = A^1_4 = 0. \tag{25}$$

Finally, one computes explicitly from (23), (19), and (15):

$$\left.\begin{array}{l}A^1_1 = g^{11}(\Gamma^1_{11})^2 + g^{44}\Gamma^1_{44}\Gamma^4_{41}\\ A^4_4 = g^{11}(\Gamma^4_{14})^2 + g^{44}\Gamma^4_{41}\Gamma^1_{44}\end{array}\right\} = \tag{26}$$

$$= -\frac{R-\alpha}{R}\cdot\frac{1}{4}\frac{\alpha^2}{R^2(R-\alpha)^2} + \frac{R}{R-\alpha}\cdot\frac{1}{4}\frac{\alpha^2}{R^4} = 0.$$

Thus, on the x_1-axis, all quantities A^α_σ vanish identically in $R(= x_1)$. Due to (21), the same holds for the t^α_σ. As remarked above in anticipation, it follows, due to the covariance of these quantities under linear transformations and due to the spherical symmetry of the field, that the t^α_σ vanish identically everywhere (outside the gravitating sphere) in the chosen reference system in all coordinates. Q.E.D.

3. This result seems to me, in any case, to be of considerable importance for our understanding of the physical nature of the gravitational field. For either we must abandon viewing the t^α_σ defined by equations (2) as the energy components of the gravitational field; but then, first of all, the significance of the "conservation laws" (cf. A. Einstein loc. cit.) would disappear, and the task would arise to securely re-establish this integral component of the foundations anew. However, if we adhere to the expressions (2), then our calculation teaches us that there are real gravitational fields (i.e., fields that cannot be "transformed away") with completely vanishing—or, more correctly, "transformable-away"—energy components; fields in which not only momentum and energy flux, but also the energy density and the analogues of Maxwellian stresses can be made to vanish for finite regions by a suitable choice of coordinate system.